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REPORT No. 888

The Deflection of A Continuous Beam Produced By The Vertical Motion of A Support Point

TURNER L. SMITH

DEPARTMENT OF THE ARMY PROJECT No. 503-06-004
ORDNANCE RESEARCH AND DEVELOPMENT PROJECT No. TB3-0118H

BALLISTIC RESEARCH LABORATORIES

ABERDEEN PROVING GROUND, MARYLAND

B A L L I S T I C   R E S E A R C H   L A B O R A T O R I E S

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THE DEFLECTION OF A CONTINUOUS BEAM PRODUCED BY THE VERTICAL MOTION OF A SUPPORT POINT

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A B E R D E E N   P R O V I N G   G R O U N D ,   M A R Y L A N D

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THE DEFLECTION OF A CONTINUOUS BEAM PRODUCED BY THE VERTICAL MOTION OF A SUPPORT POINT

ABSTRACT

The elastic curve for a uniform continuous beam, simply restrained to have zero deflection at four or more equidistant points, satisfies a second order linear finite difference equation. The general solution for the elastic curve is found to be the sum of a decreasing wave and an increasing wave. By a combination of these, the elastic curve produced by moving any one support point is obtained, including the cases where the support point which is moved is at one end or near one end. These results are useful in obtaining a logical procedure for correcting supersonic nozzles.

support points, the right member of equation (1.1) is a linear function of x. Hence,

Theorem 1. The deflection y is continuous with continuous slope and curvature (y' and y''), and each segment between support points is a cubic.

DERIVATION OF THE DIFFERENCE EQUATION

Let the deflection of the beam be zero at four successive support points, P_1, P_2, P_3, P_4, and take individual axis systems (x_i, y_i) for the three included segments of the elastic curve.

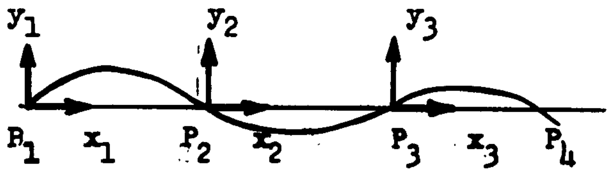


Figure 1.

Then, since each segment is a cubic which vanishes at its origin, the equations of these segments are

$$Y_i = A_i x_i^3 + B_i x_i^2 + C_i x_i, \ i = 1, 2, 3$$

Continuity at P_2, P_3, and P_4 gives three equations found by letting $x_i = 1$.

(2.1)
$$A_1 + B_1 + C_1 = 0$$
$$A_2 + B_2 + C_2 = 0$$
$$A_3 + B_3 + C_3 = 0$$

Continuity of y' and y'' at P_2 and P_3 give

(2.2)
$$3A_1 + 2B_1 + C_1 = C_2$$
$$3A_2 + 2B_2 + C_2 = C_3$$
$$6A_1 + 2B_1 = 2B_2$$
$$6A_2 + 2B_2 = 2B_3$$

These seven equations for nine unknowns leave two degrees of freedom. The elastic curve would be completely determined, for example, if the slopes were prescribed at both ends P_1 and P_4. From seven equations we can eliminate six unknowns and obtain a single equation in the remaining three. In this manner, it is found as a consequence of the seven equations (2.1) and (2.2) that

$$(2.3)\qquad \begin{aligned} A_1 + 4A_2 + A_3 &= 0 \\ B_1 + 4B_2 + B_3 &= 0 \\ C_1 + 4C_2 + C_3 &= 0 \end{aligned}$$

Since the moment is

$$M_i = EIy_i'' = EI(6A_ix_i + 2B_i)$$

and hence the moment at $x_i = 0$ is $M_i = 2B_iEI$, it follows that

$$M_1 + 4M_2 + M_3 = 0$$

which is the well-known "three-moment equation". Similarly, by letting $x_i = 1$, we get

$$M_2 + 4M_3 + M_4 = 0$$

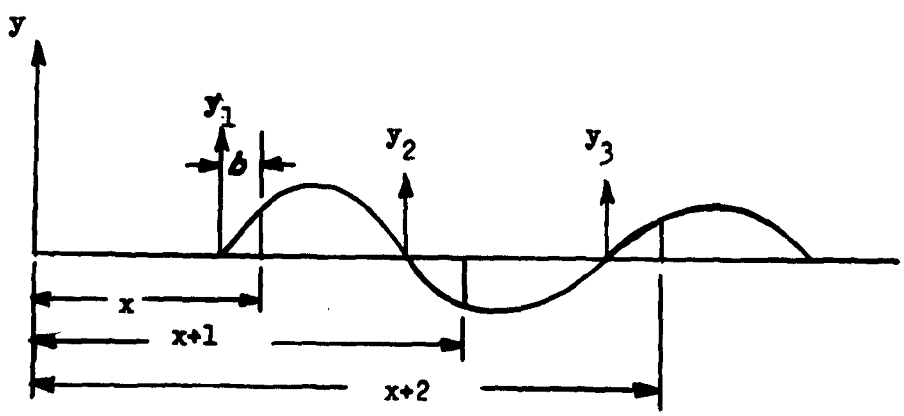


Figure 2

If we form the function

$$y(x+2) + 4y(x+1) + y(x) \tag{2.4}$$

involving heights at corresponding points in three successive segments, this may be written

$$y_3(b) + 4y_2(b) + y_1(b) = (A_3 + 4A_2 + A_1)b^3 + (B_3 + 4B_2 + B_1)b^2 + (C_3 + 4C_2 + C_1)b$$

which vanishes on account of equations (2.3) for all values of b in the range $0 \leq b \leq 1$. Hence, the expression (2.4) is zero for any x in the interval $P_1 - P_2$.

Since y(x) is of class C", the first and second derivative of the expression (2.4) also vanishes. Moreover, if the elastic curve has zero deflection at more than four successive support points, it is evident that the same expressions vanish everywhere on any portion of the elastic curve in this region. Hence, we get

Theorem 2. If a uniform continuous beam is simply supported at the support points $x = a, a+1, \ldots, a+n$, with $n \geq 3$, and has zero deflection at all these support points, then the deflection y(x) satisfies the finite difference equation

$$y(x+2) + 4y(x+1) + y(x) = 0 \tag{2.5}$$

in the closed interval $a \leq x \leq a+n$.

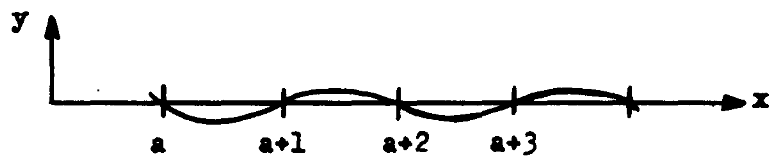


Figure 3

Since y(x) is of class C" in this interval, equation (2.5) may be differentiated twice to yield the results:

Theorem 2a. The functions y', y" and hence the slopes, curvatures and moments satisfy also the same difference equation.

SOLUTION OF THE DIFFERENCE EQUATION

It is intuitively evident that the elastic curve of n (≥ 3) segments through n + 1 support points where the deflection is zero is uniquely determined by prescribing slopes at both ends; hence, there is a two-parameter family of such elastic curves. Let us investigate how this appears mathematically. The difference equation

$$y(x+2) + 4y(x+1) + y(x) = 0$$

may be solved by letting

$$y(x) = \beta^x,$$

whence

$$y(x+1) = \beta^{x+1} = \beta^x \beta = \beta y(x)$$

$$y(x+2) = \beta^{x+2} = \beta^x \beta^2 = \beta^2 y(x)$$

The difference equation then becomes

$$(3.1) \quad \beta^2 + 4\beta + 1 = 0$$

which has the two roots

$$\beta = -2 + \sqrt{3} = -1/(2 + \sqrt{3}) = -0.26795$$

$$1/\beta = -2-\sqrt{3} = -1/(2 - \sqrt{3}) = -3.7320$$

The solution of a linear homogeneous difference equation is not unique. For it is easy to check that if P(x) is any periodic function of period unity,

$$P(x+1) = P(x)$$

and if $y = f(x)$ is a solution of the difference equation

$$y(x+2) + 4y(x+1) + y(x) = 0$$

then

$y = P(x) \,.\, f(x)$ is also a solution. (This is the analog to the theorem in linear homogeneous differential equations that any constant times a solution is a solution).

When this result is applied to the continuous beam problem in any interval, $a \leq x \leq a + n$, we conclude that the solutions $y(x) = P(x)\beta^x$ or $P(x)/\beta^x$ must be cubics in any segment between successive intergers.

We therefore get the result:

Theorem 3. <u>In any interval of n segments ($n \geq 3$) of an elastic curve where the deflection vanishes at every support point, the deflection is a linear combination of an increasing "wave", $y(x+1) = \beta y(x)$, and a</u>

decreasing "wave" with $y(x+1) = (1/\beta)y(x)$.

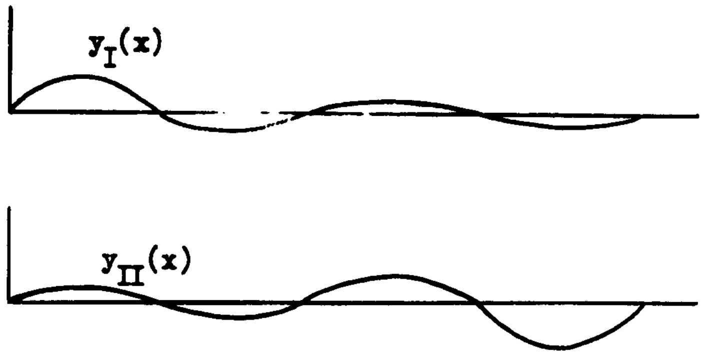


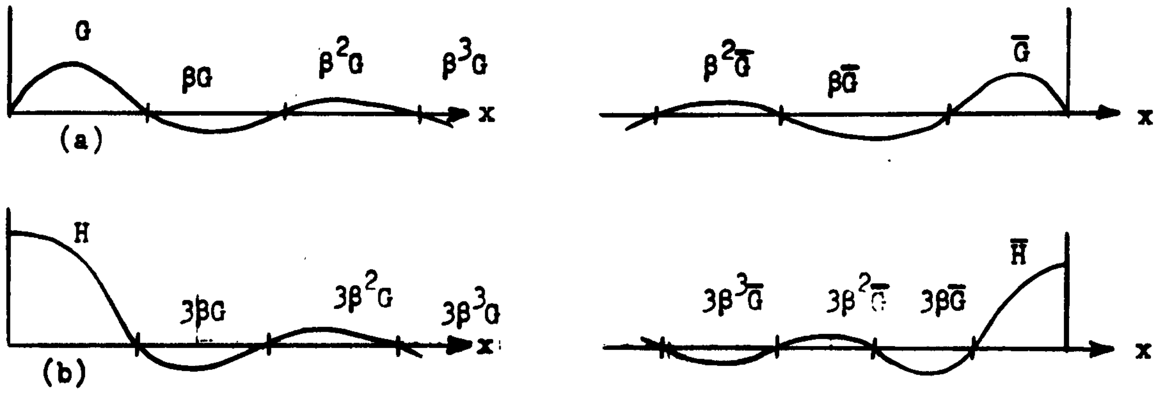
Figure 4

General solution is $C_I y_I(x) + C_{II} y_{II}(x)$.

Since $|\beta| = 1/4$ approximately, the decreasing "wave" decays to about 0.1% of its initial amplitude in five intervals, and hence can be considered to vanish for most engineering applications in five or less intervals.

THE TWO FUNDAMENTAL SOLUTIONS

It is convenient to make use of two solutions for the semi-infinite beam with zero deflections at the support points $x = 1,2,3, \ldots$, and decaying toward infinity. The first fundamental solution has zero deflection and unit slope at $x = 0$ (Figure 5a), while the second has unit deflection and zero slope at the origin (Figure 5b). It will be found that the solutions of particular problems can be built up from these fundamental solutions and their reflections, (Figures 5c and 5d.).



Figures 5

The first fundamental solution satisfies the difference equation over the entire range $0 < x < \infty$ and hence is completely determined after we find the function G for the first segment $0 \leq x \leq 1$. The succeeding segments are the curves β G, β^2 G, etc. Thus, the deflection, slope, and curvature at the right end of G are β times their initial values to give continuity of these functions at x = 1.

Hence the cubic for G,

$$y = Ax^3 + Bx^2 + Cx + D,$$

satisfies the initial conditions

$$y(0) = D = 0$$

$$y'(0) = C = 1$$

and the continuity conditions at x = 1 (after inserting values of C,D)

$$A + B + 1 = 0$$

$$3A + 2B + 1 = \beta$$

$$6A + 2B = 2B\beta$$

These three equations are linearly dependent; the solution of any pair gives

$$G = (\sqrt{3} - 1)\, x^3 - \sqrt{3}\, x^2 + x \tag{4.1}$$

The second fundamental solution (Figure 5b) satisfies the difference equation to the right of x = 1, so that the elastic curve to the right of x = 1 is known to be - g G, where - g is the slope at x = 1. Thus for the function H,

$$y = Ax^3 + Bx^2 + Cx + D$$

we have from initial conditions

$$y(0) = D = 1$$

$$y'(0) = C = 0$$

From continuity with -g G at x = 1 (inserting values of C,D)

$$y(1) = A+B+1 = -gG(0) = 0$$
$$y'(1) = 3A+2B = -gG'(0) = -g$$
$$y''(1) = 6A+2B = gG''(0) = 2g\sqrt{3}$$

By solving these we find that

(4.2) $$h = (3\sqrt{3} - 4)x^3 - (3\sqrt{3} - 3)x^2 + 1$$

and that

(4.3) $$g = 6 - 3\sqrt{3} = -3\beta$$

To find the reflected functions $\bar{G}$ and $\bar{H}$ referred to axes at their left ends for convenience, it is only necessary to replace x by (1 - x) in 4.1 and 4.2.

We get

(4.4) $$\bar{G}(x) = G(1-x) = (-\sqrt{3} + 1)x^3 + (2\sqrt{3} - 3)x^2 + (2 - \sqrt{3})x$$

(4.5) $$\bar{H}(x) = H(1-x) = -(3\sqrt{3} - 4)x^3 + (6\sqrt{3} - 9)x^2 + (6 - 3\sqrt{3})$$

Putting the coefficients in decimal form, we have the equations

$$G(x) = 0.73205x^3 - 1.73205x^2 + x$$

$$H(x) = 1.19615x^3 - 2.19615x^2 + 1$$

$$\beta = -0.26795$$

$$g = 0.80385 = -H'(1)$$

Larger graphs of the functions G and H are given at the end of this report.

APPLICATION TO SEMI-INFINITE BEAM

A. End of semi-infinite beam given a deflection h and a slope λ

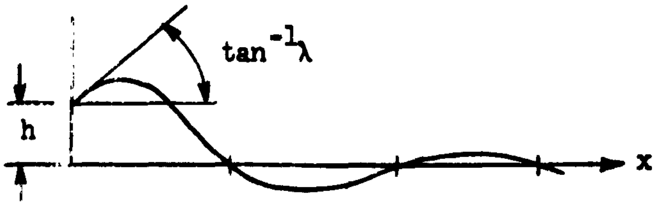


Figure 6

The function

$$y = hH(x) + \lambda G(x)$$

is easily seen to be the solution here, where H, G are extended beyond $x = 1$, as in Figure 5a and b. The slope at $x = 1$ is $hH'(1)+\lambda G'(1) = 3\beta h + \beta\lambda$, hence, the second segment of Fig. 6 is $(3h+\lambda)\beta G$, and the nth segment is $(3h+\lambda)\beta^{n-1}G$.

When any other jack than the end jack is moved in the semi-infinite plate, the plate shape is the sum of a zeroth order approximation, curve (0), and a first order correction, curve (1). Curve (0) is the elastic curve which would be obtained in an infinite plate extending over the range $-\infty < x < \infty$; curve (1) is the reflection of this curve at the fixed end $x = 0$. In the case of a finite length plate, treated in the next section, the zeroth approximation, curve (0), is again the elastic curve for an infinite plate; to this must be added an infinite number of reflections of curve (0), reflected from both ends of the actual finite plate.

B. End of beam fixed at zero slope, support point at $x = 1$ raised by h units.

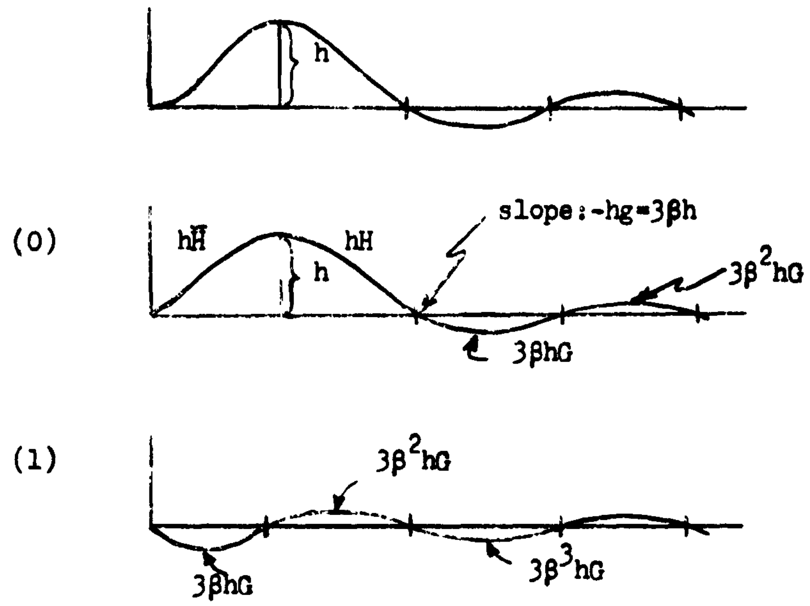


Figure 7

Curve (0) is composed of the fundamental solution H multiplied by the ordinate h starting at $x = 1$ and decaying to the right; the reflection function $h\bar{H}$ extends to the left of $x = 1$ to the origin.

Curve (1) is $-\lambda G$ starting from the origin with a slope opposite that of curve (1), where $\lambda = hg$.

The sum of curves (0) and (1) is the complete solution.

Referred to axes at the left end of each segment, the equations of the successive segments in the solution are

$$y_1 = h\left[\bar{H}(x_1) + 3\beta G(x_1)\right]$$

$$y_2 = h\left[H(x_2) + 3\beta^2 G(x_2)\right]$$

$$y_3 = h\left[3(1+\beta^2)\beta G(x_3)\right]$$

$$y_4 = \beta y_3,$$

$$y_n = \beta^{n-3} y_3 = h\,3\beta^{n-2}(1+\beta^2)G, \text{ for } n \geq 3.$$

Clearly curve (1) can be considered to be the left wave $h\bar{H}$ of curve (0) "reflected" at $x = 0$.

SEMI-INFINITE BEAMS

C. Left end of beam fixed, support at x = 2 raised by height h.

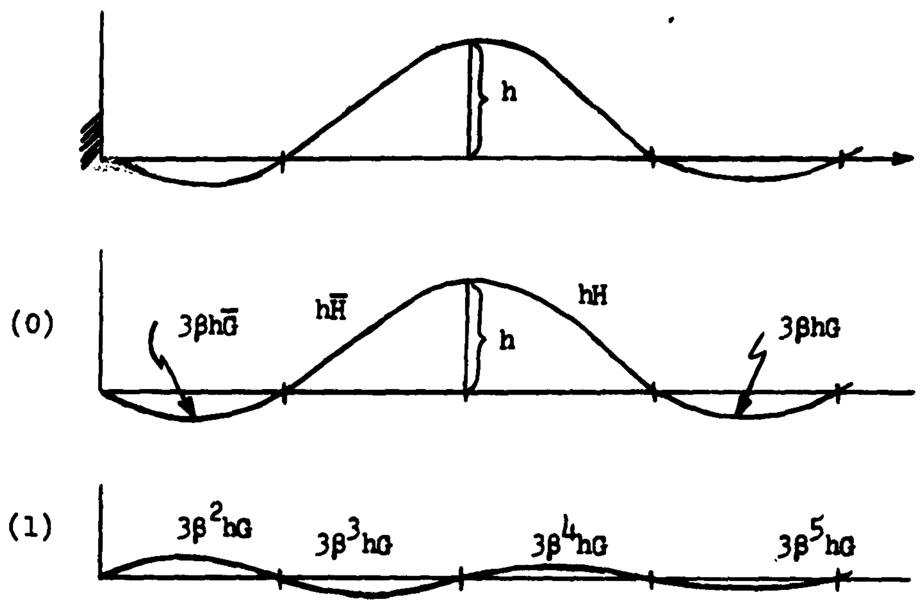


Figure 8

The solution is the sum of curves (0) and (1).

Again, curve (1) can be considered as the reflection of the "wave" to the left of the origin in figure (8).

Successive segments referred to their own axis systems

$$y_1 = 3\beta h(\bar{G}+\beta G)$$

$$y_2 = h(\bar{H}+3\beta^3 G)$$

$$y_3 = h(H+3\beta^4 G)$$

$$y_4 = 3h\beta(1+\beta^4)G$$

$$y_n = \beta^{n-4} y_4, \ n = 5, 6, \ldots\ldots$$

D. <u>Support at $x = 3$ raised.</u>

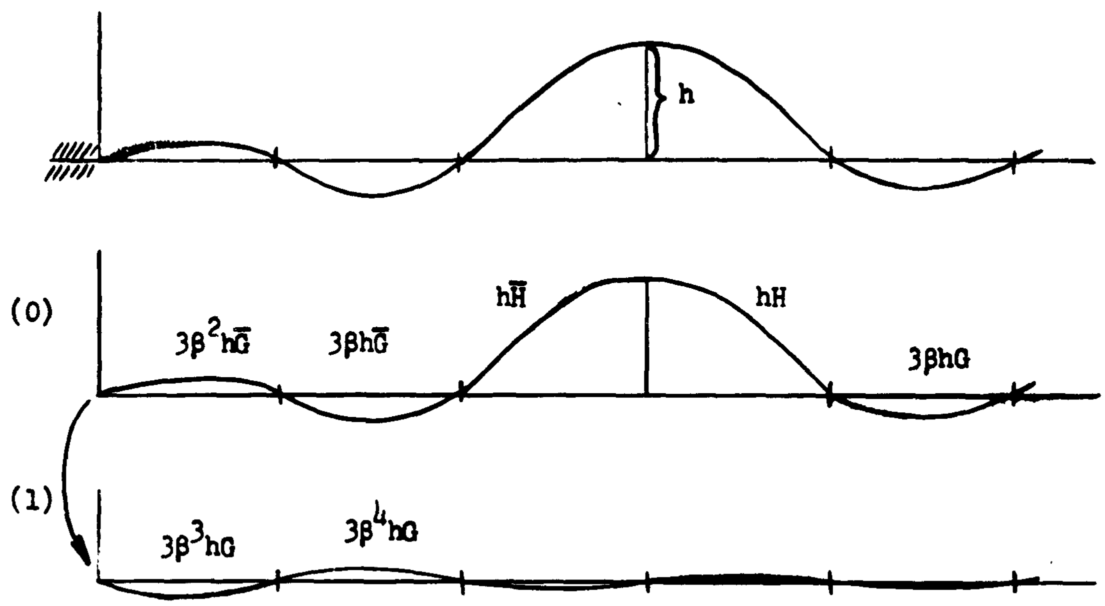


Figure 9

Successive segments

$$y_1 = 3h(\beta^2\bar{G}+\beta^3 G)$$

$$y_2 = 3h(\beta\bar{G}+\beta^4 G)$$

$$y_3 = h(\bar{H}+\beta^5 G)$$

$$y_4 = h(H+\beta^6 G)$$

$$y_5 = 3\beta h(1+\beta^6)G,$$

$$y_n = \beta^{n-5} y_5, \; n = 6, 7, \ldots\ldots$$

FINITE BEAM PROBLEMS WITH FIXED ENDS

a. Two spans

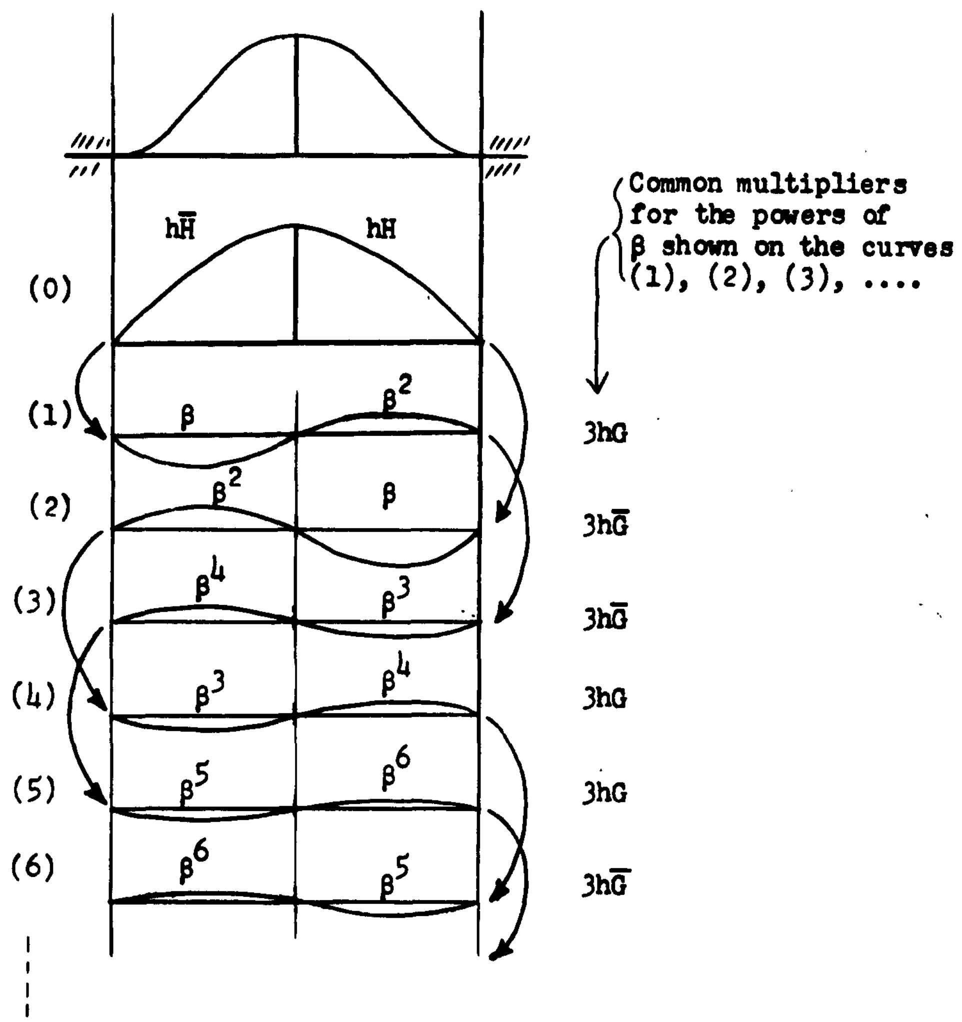


Figure 10

Curve 1 is the first reflection from the left of curve 0; the sum of these satisfies the zero slope condition on the left, but not on the right. When the reflections 2 and 3 from the right of curves 0 and 1 are added on, the zero slope condition is satisfied on the right, but not on the left. When the infinite series of reflections back and forth are added, the exact solutions are obtained.

$$y_1 = h\bar{H}(x_1) + 3\beta h\left[(1+\beta^2+\beta^4+\beta^6+ \ldots)G(x_1)+ (\beta+\beta^3+\beta^5+ \ldots)\bar{G}(x_1)\right]$$

$$= h\bar{H}(x_1) + 3\beta h(G+\beta\bar{G})/(1-\beta^2)$$

This illustrates a property which will be found to be true for fixed end beams with any number of spans:

Theorem: The four terms in the reflections 1, 2, 3, and 4 for any span will always be the first terms in four geometric series of ratio β^{2n} where n is the number of spans. Since

$$A + Ar + Ar^2 + Ar^3 + \ldots = A/(1-r), \text{ for } |r| < 1$$

it is only necessary to add the zeroth order curve plus $1/(1-\beta^{2n})$ times the sum of the first four reflection curves 1, 2, 3, 4 in order to get the exact solution.

The second segment of the beam is found by adding its zeroth order term $+1/(1-\beta^4)$ times the terms in the reflections 1, 2, 3, and 4. In terms of its own coordinate system.

$$y_2 = hH(x_2) + 3\beta h\left[(\beta+\beta^3)G(x_3) + (1+\beta^2)\bar{G}(x_2)\right]/(1-\beta^4)$$

or

$$y_2 = hH(x_2) + 3\beta h\left[\beta G(x_2)+\bar{G}(x_2)\right]/(1-\beta^2).$$

b. Three-Span beam

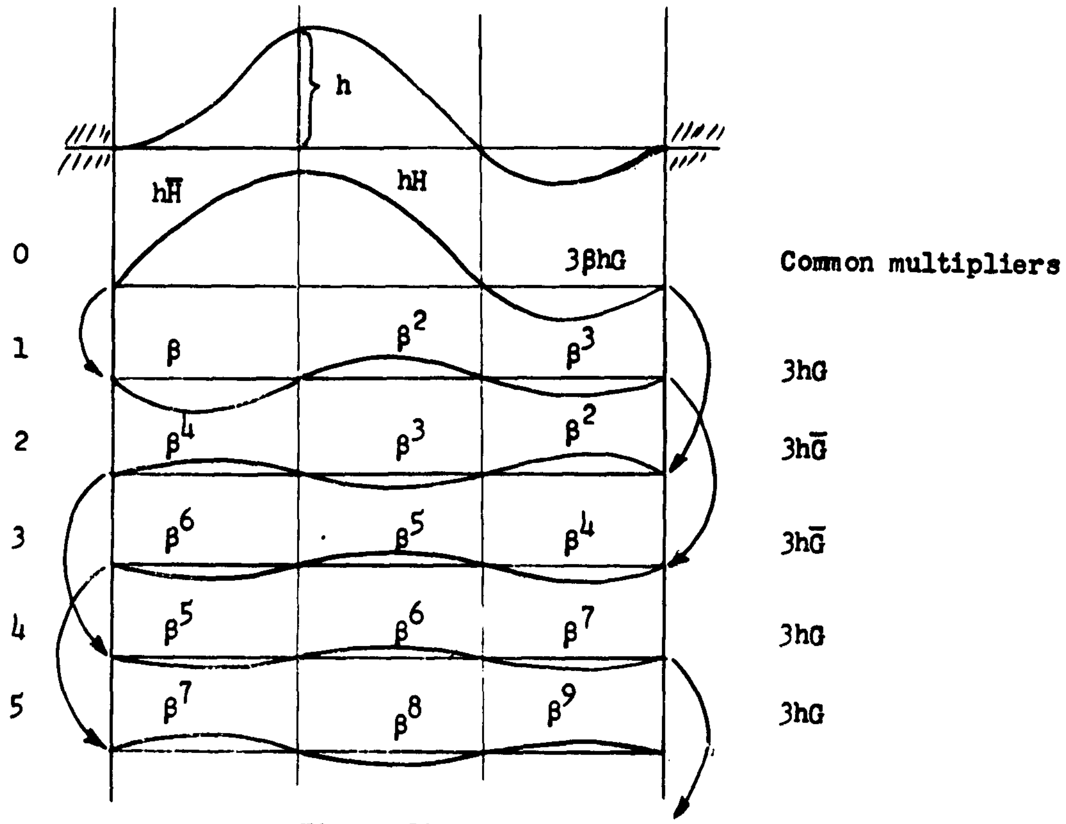


Figure 11

In this diagram, common multipliers for the terms in the first four reflection are placed at the right while only the proper powers of β are placed on the corresponding segments.

The solution is

$$y_1 = h\bar{H}(x_1) + 3\beta h \left[(1+\beta^4)G(x_1) + (\beta^3+\beta^5)\bar{G}(x_1)\right] / (1-\beta^6)$$

$$y_2 = hH(x_2) + 3\beta h \left[(\beta+\beta^5)G(x_2)+(\beta^2+\beta^4)\bar{G}(x_2)\right] / (1-\beta^6)$$

$$\begin{aligned} y_3 &= 3\beta hG(x_3) + 3\beta h \left[(\beta^2 + \beta^6)\, G(x_3) + (\beta + \beta^3)\, \bar{G}(x_3)\right] / (1-\beta^6) \\ &= 3\beta h \left[(1+\beta^2)\, G(x_3) + (\beta + -\beta^3)\bar{G}(x_3)\right] / (1-\beta^6) \\ &= 3\beta h(1+\beta^2) \left[G(x_3) + \beta G(x_2)\right]/(1-\beta^6) \end{aligned}$$

c. Five-span Beam with Prescribed Initial Height and Slope

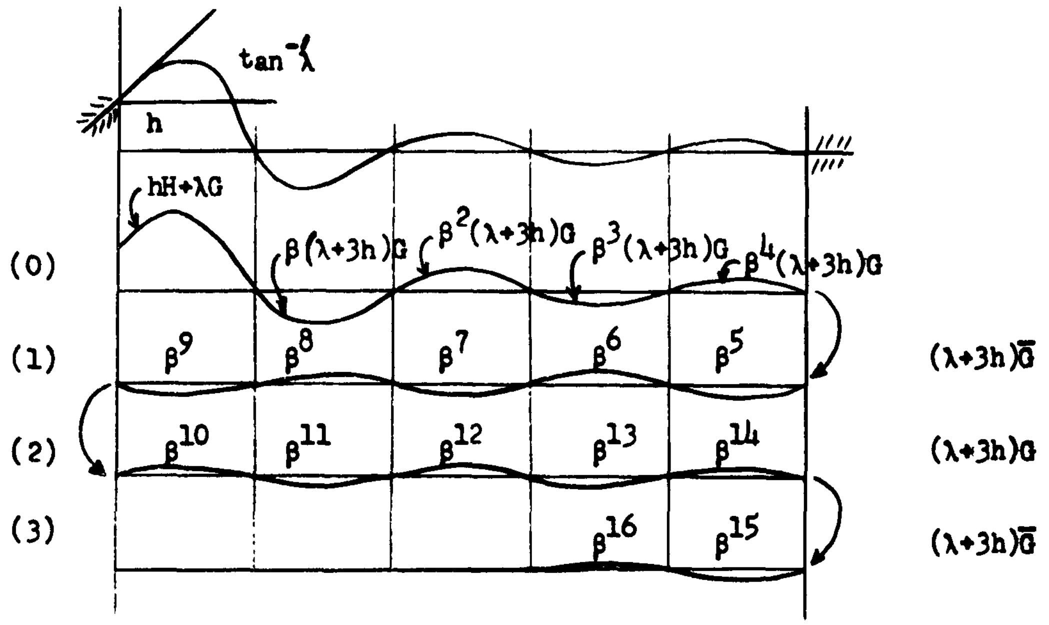


Figure 12

The zeroth order curve shown satisfies left end boundary conditions, hence only two reflections, curves 1 and 2, need be considered to get the first terms of the infinite series (geometrical) and hence to obtain the exact solution.

For the 5 span beam shown:

$$y_1 = hH + \lambda G + \beta(\lambda+3h)\left[\beta^9 G + \beta^8 \bar{G}\right]$$

$$y_2 = \beta(\lambda+3h)G + \beta(\lambda+3h)\left[\beta^{10} G + \beta^7 \bar{G}\right]/(1-\beta^{10}) = +\beta(\lambda+3h)\left[G + \beta^7 \bar{G}\right]/(1-\beta^{10})$$

$$y_3 = \beta\,(\lambda+3h)\left[\beta G + \beta^6 \bar{G}\right] / (1-\beta^{10})$$

$$y_4 = \beta(\lambda+3h)\left[\beta^2 G + \beta^5 \bar{G}\right] / (1-\beta^{10})$$

$$y_5 = \beta(\lambda+3h)\left[\beta^2 G + \beta^5 \bar{G}\right]/(1-\beta^{10})$$

REFLECTION FROM SIMPLY-SUPPORTED END

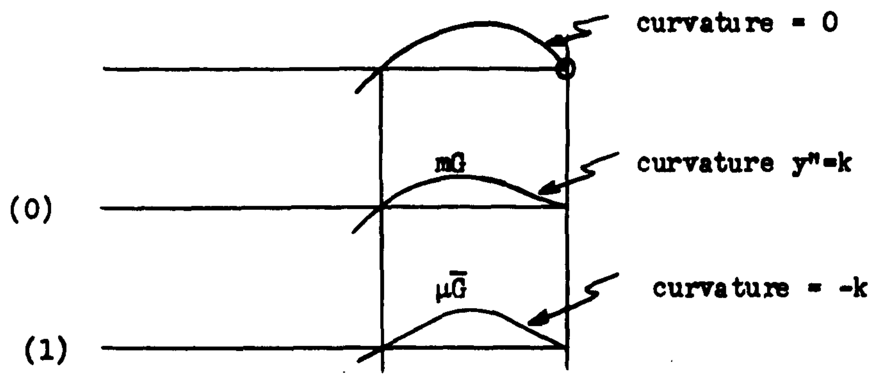


Figure 13

Suppose the zeroth order approximation ends with the terms mG, and the reflection curve 1 ends with the terms $\mu\bar{G}$; then in order that the sum of curves 0 and 1 satisfy the boundary condition $y'' = 0$, we must have

$$mG''(1)+\mu\bar{G}''(1)=0$$

$$m(6\sqrt{3} - 6 - 2\sqrt{3}) + \mu(-6\sqrt{3} + 6 + 4\sqrt{3} - 6) = 0$$

Therefore $\mu = + k(\text{end})/2\sqrt{3}$

$$= (4\sqrt{3} - 6)m/2\sqrt{3} = (2 - \sqrt{3})m = -\beta m$$

Theorem: <u>The reflection on the right hand end starts with $k\bar{G}/2\sqrt{3}$ where k is the curvature at the right on the zero order approximation. If the right of the zeroth order curve is mG, then the right end of the reflection curve is $(-m\beta\bar{G})$. The reflection from a simply supported end is with no change in sign of y, since $-\beta$ is positive. The same theorem holds at a simply supported left end, with the words left and right and the symbols G and $\bar{G}$ interchanged</u>.

As a consequence, if the zeroth order approximation ends in a multiple of $\bar{G}$ (at the left) or a multiple of G (on the right), the exact solution has twice the end slope of the zeroth approximation.

The same principle of multiplying by $-\beta$ instead of by β holds, of course, for all the infinite series of reflections needed in a finite beam with pin-supported (simply supported) ends.

See Fig. 14 for an example of a 3-span beam with the left end pin-connected and the right end fixed.

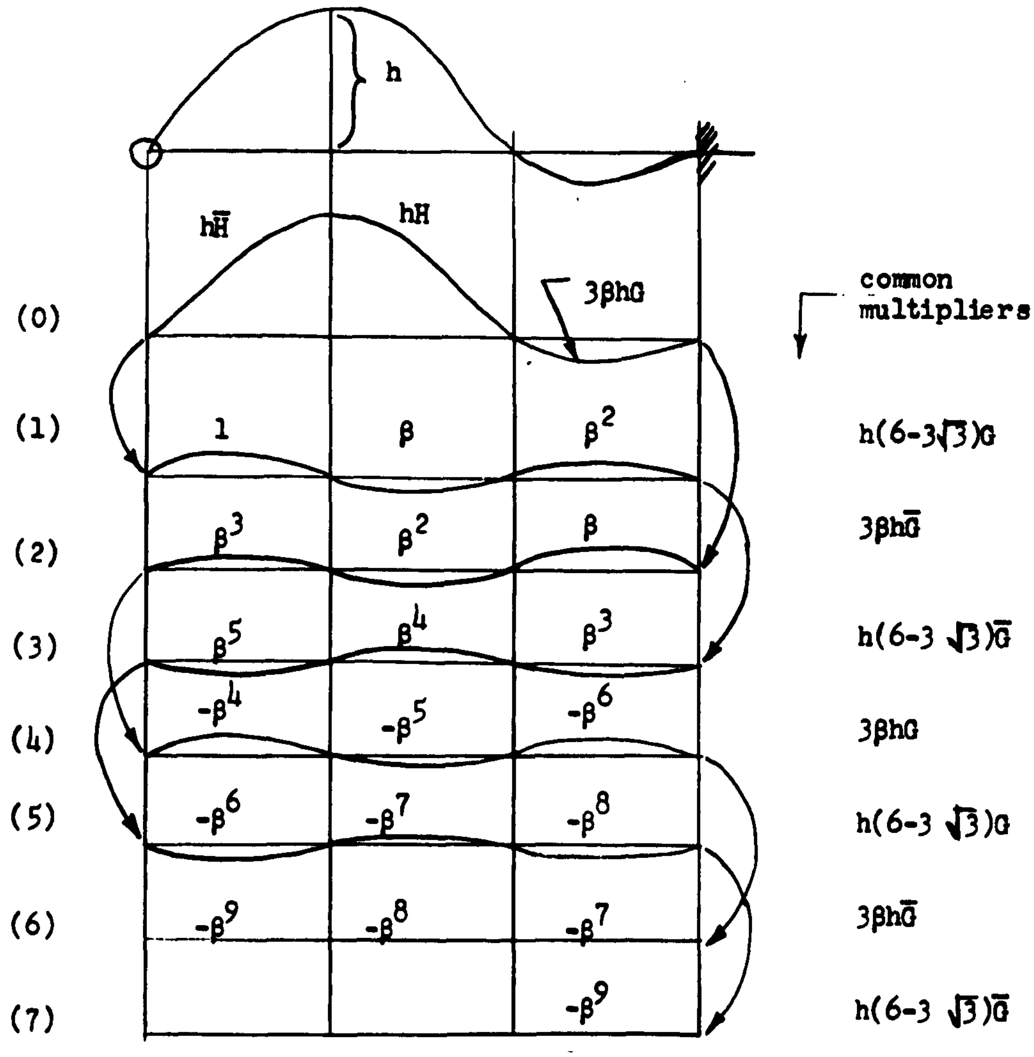


Figure 14

$$y_1 = h\bar{H} + (6-3\sqrt{3})\,h\left[(1-\beta^6+\beta^{12}-\ldots)G+(\beta^5-\beta^{11}+\beta^{17}-\ldots)\right]$$

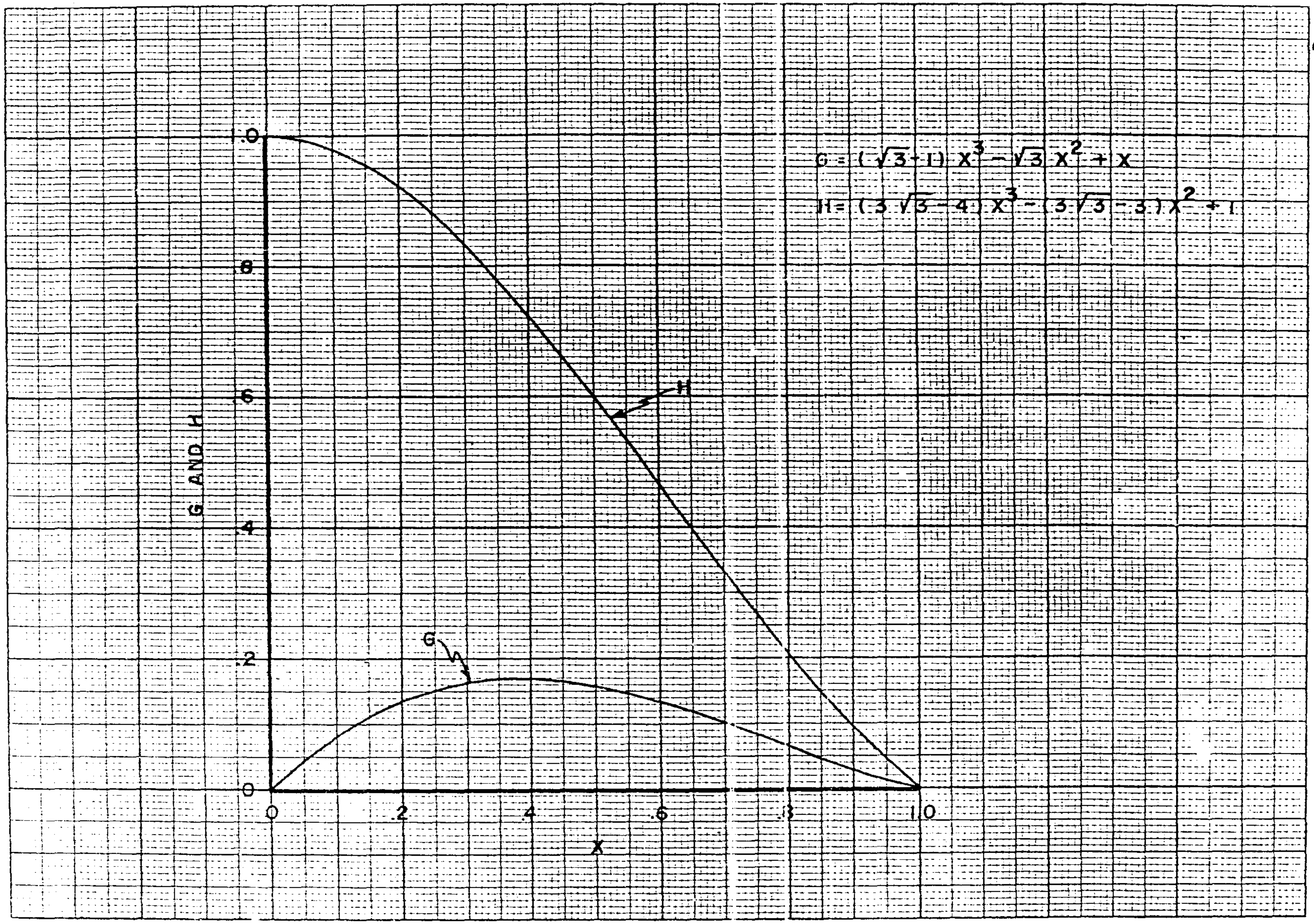
$$+3\beta h\left[(\beta^3-\beta^9+\beta^{15}-\ldots)+(\beta^4-\beta^{10}+\beta^{16}-\ldots)\right]$$

$$= h\bar{H}+(6-3\sqrt{3})h(G+\beta^5\bar{G})/(1+\beta^6)+3\beta h(\beta^3\bar{G}+\beta^4 G)/(1+\beta^6)$$

Again the first four reflection curves are enough, but the geometric series are alternating with the ratio $-\beta^6$; hence their first terms must be divided by $1+\beta^6$ to obtain their sums.

Turner L. Smith

TURNER L. SMITH



G = (√3 - 1) X³ - √3 X² + X
H = (3 √3 - 4) X³ - (3 √3 - 3) X² + 1
H
G
G AND H
1.0
.8
.6
.4
.2
0
0
.2
.4
.6
.8
1.0
X

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